Supplementary Material

1 Notations

In this paper, we use the following notations. Other notations will be clearly defined when they appear.

- $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and natural numbers, respectively.
- Bold letters, e.g., $\mathbf{b}$, $\mathbf{f}$, $\mathbf{u}$, denote vectors.
- Capital letters, e.g., A, F, U, denote matrices.
- $\mathbf{f}_i$ denotes the ith entry of the vector $\mathbf{f}$.
- $A_{i,j}$ denotes the (i,j)th entry of the matrix A.
- I_n denotes the identity matrix of size n-by-n.
- $\mathbf{0}$ denotes the zero vector of an appropriate length.
- $\text{vec}(A) = (A_{1,1}, \ldots, A_{m,1}, A_{1,2}, \ldots, A_{m,2}, \ldots, A_{1,n}, \ldots, A_{m,n})^\top$ denotes the vectorization of $A \in \mathbb{R}^{m\times n}$.

2 Mathematical Details

Following [1], we provide the proofs of the propositions, lemmas, and theorems stated in the main paper below.

Proposition 1. *The minimization problem (5) has a unique solution.*

Proof. The composition of a convex function with an affine map is convex, and the sum of a convex function and a strictly convex function is strictly convex. Hence, (5) is strictly convex and admits a unique minimizer. $\square$

Theorem 1. *Let $\{(\mathbf{u}^{(k+1)}, \mathbf{d}^{(k+1)}, \mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})\}_{k\in\mathbb{N}}$ be sequences given by the iteration scheme (15) and let $\tilde{\mathbf{u}}$ be the unique solution to the minimization problem (5). Then*

$$\lim_{k\to\infty} \mathbf{u}^{(k)} = \tilde{\mathbf{u}}.$$

Proof. By Proposition 1, there exists a unique solution $\tilde{\mathbf{u}}$ to the minimization problem (5). From the equivalences of the reformulations (6) and (7), we conclude that $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is the unique solution to the minimization problem (7) that satisfies $\tilde{\mathbf{d}} = \mathbf{f} - \tilde{\mathbf{u}}$, $\tilde{\mathbf{x}} = D_x\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}} = D_y\tilde{\mathbf{u}}$. By applying Lemma 1 to (7) with

$$E(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1 + \mu\|\mathbf{d}\|_1 + \alpha\|\mathbf{u} - \mathbf{f}\|_2^2,$$

and

$$H(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y}) = \lambda\Big(\big\|\mathbf{d} - (\mathbf{f} - \mathbf{u})\big\|_2^2 + \big\|\mathbf{x} - D_x\mathbf{u}\big\|_2^2 + \big\|\mathbf{y} - D_y\mathbf{u}\big\|_2^2\Big),$$

we obtain

$$\lim_{k\to\infty} \big\|\mathbf{d}^{(k)} - (\mathbf{f} - \mathbf{u}^{(k)})\big\|_2^2 = 0,\ \lim_{k\to\infty} \big\|\mathbf{x}^{(k)} - D_x\mathbf{u}^{(k)}\big\|_2^2 = 0,\ \text{and}\ \lim_{k\to\infty} \big\|\mathbf{y}^{(k)} - D_y\mathbf{u}^{(k)}\big\|_2^2 = 0. \tag{1}$$

By applying Lemma 3 to $E(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y})$, we obtain

$$\lim_{k\to\infty} \Big\|\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}\Big\| = 0.$$

By multiplying the sequence by D_x and D_y, respectively, it follows that

$$\lim_{k\to\infty} \Big\|D_x\mathbf{u}^{(k+1)} - D_x\mathbf{u}^{(k)}\Big\| = 0\ \text{ and }\ \lim_{k\to\infty} \Big\|D_y\mathbf{u}^{(k+1)} - D_y\mathbf{u}^{(k)}\Big\| = 0. \tag{2}$$

From (1), (2) and the triangle inequality, we obtain

$$\lim_{k\to\infty} \Big\|\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}\Big\| = 0,\ \lim_{k\to\infty} \Big\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\Big\| = 0,\ \text{and}\ \lim_{k\to\infty} \Big\|\mathbf{y}^{(k+1)} - \mathbf{y}^{(k)}\Big\| = 0.$$

To prove the convergence $\lim_{k\to\infty} \mathbf{u}^{(k)} = \tilde{\mathbf{u}}$, by applying Lemma 2, it suffices to show that the sequences $\{(\mathbf{u}^{(k)}, \mathbf{d}^{(k)}, \mathbf{x}^{(k)}, \mathbf{y}^{(k)})\}_{k\in\mathbb{N}\setminus\{0\}}$, $\{\mathbf{b}_1^{(k)}\}_{k\in\mathbb{N}\setminus\{0\}}$, $\{\mathbf{b}_2^{(k)}\}_{k\in\mathbb{N}\setminus\{0\}}$, and $\{\mathbf{b}_3^{(k)}\}_{k\in\mathbb{N}\setminus\{0\}}$ are bounded. From the optimality condition of (15), we have

$$\begin{aligned}
\mu\|\mathbf{d}^{(k+1)}\|_1 + \lambda\|\mathbf{d}^{(k+1)} - (\mathbf{f} - \mathbf{u}^{(k+1)}) - \mathbf{b}_1^{(k)}\|_2^2 &\le \mu\|\mathbf{d}\|_1 + \lambda\|\mathbf{d} - (\mathbf{f} - \mathbf{u}^{(k+1)}) - \mathbf{b}_1^{(k)}\|_2^2 \text{ for all } \mathbf{d} \in \mathbb{R}^{mn},\\
\|\mathbf{x}^{(k+1)}\|_1 + \lambda\|\mathbf{x}^{(k+1)} - D_x\mathbf{u}^{(k+1)} - \mathbf{b}_2^{(k)}\|_2^2 &\le \|\mathbf{x}\|_1 + \lambda\|\mathbf{x} - D_x\mathbf{u}^{(k+1)} - \mathbf{b}_2^{(k)}\|_2^2 \text{ for all } \mathbf{x} \in \mathbb{R}^{mn},\\
\|\mathbf{y}^{(k+1)}\|_1 + \lambda\|\mathbf{y}^{(k+1)} - D_y\mathbf{u}^{(k+1)} - \mathbf{b}_3^{(k)}\|_2^2 &\le \|\mathbf{y}\|_1 + \lambda\|\mathbf{y} - D_y\mathbf{u}^{(k+1)} - \mathbf{b}_3^{(k)}\|_2^2 \text{ for all } \mathbf{y} \in \mathbb{R}^{mn}.
\end{aligned}$$

Thus, from the property of the soft shrinkage function defined in (9), we have

$$\begin{aligned}
\mathbf{d}^{(k+1)} &= \operatorname{shrink}_1\left(\mathbf{f} - \mathbf{u}^{(k+1)} + \mathbf{b}_1^{(k)}, \frac{\mu}{2\lambda}\right),\\
\mathbf{x}^{(k+1)} &= \operatorname{shrink}_1\left(D_x\mathbf{u}^{(k+1)} + \mathbf{b}_2^{(k)}, \frac{1}{2\lambda}\right),\\
\mathbf{y}^{(k+1)} &= \operatorname{shrink}_1\left(D_y\mathbf{u}^{(k+1)} + \mathbf{b}_3^{(k)}, \frac{1}{2\lambda}\right).
\end{aligned}$$

These three equations together with (16) give

$$\begin{aligned}
\mathbf{b}_1^{(k+1)} &= \operatorname{cut}\left(\mathbf{f} - \mathbf{u}^{(k+1)} + \mathbf{b}_1^{(k)}, \frac{\mu}{2\lambda}\right),\\
\mathbf{b}_2^{(k+1)} &= \operatorname{cut}\left(D_x\mathbf{u}^{(k+1)} + \mathbf{b}_2^{(k)}, \frac{1}{2\lambda}\right),\\
\mathbf{b}_3^{(k+1)} &= \operatorname{cut}\left(D_y\mathbf{u}^{(k+1)} + \mathbf{b}_3^{(k)}, \frac{1}{2\lambda}\right),
\end{aligned}$$

where the cut function is defined as

$$\operatorname{cut}(\mathbf{v}, \gamma) = \mathbf{v} - \operatorname{shrink}_1(\mathbf{v}, \gamma) = \Big[\operatorname{sign}(\mathbf{v}_i)\min\{|\mathbf{v}_i|, \gamma\}\Big]_{i=1}^{mn}.$$

It is clear that $\|\operatorname{cut}(\mathbf{v}, \gamma)\|_\infty \le \gamma$. In particular,

$$\|\mathbf{b}_1^{(k+1)}\|_\infty \le \frac{\mu}{2\lambda}, \ \|\mathbf{b}_2^{(k+1)}\|_\infty \le \frac{1}{2\lambda} \ \text{ and } \ \|\mathbf{b}_3^{(k+1)}\|_\infty \le \frac{1}{2\lambda}.$$

Furthermore, the optimality condition of (15) implies

$$E(\mathbf{u}^{(k+1)}, \mathbf{d}^{(k+1)}, \mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}) + H^{(k)}(\mathbf{u}^{(k+1)}, \mathbf{d}^{(k+1)}, \mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}) \le E(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y}) + H^{(k)}(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y})$$

holds true for all $(\mathbf{u}, \mathbf{d}, \mathbf{x}, \mathbf{y})$. In particular, we consider the case $\mathbf{u} = \mathbf{d} = \mathbf{x} = \mathbf{y} = \mathbf{0}$. From the nonnegativity of $H^{(k)}$, we obtain

$$\begin{aligned}
&\|\mathbf{x}^{(k+1)}\|_1 + \|\mathbf{y}^{(k+1)}\|_1 + \mu\|\mathbf{d}^{(k+1)}\|_1 + \alpha\|\mathbf{u}^{(k+1)} - \mathbf{f}\|_2^2\\
&= E(\mathbf{u}^{(k+1)}, \mathbf{d}^{(k+1)}, \mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})\\
&\le E(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) + H^{(k)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) - H^{(k)}(\mathbf{u}^{(k+1)}, \mathbf{d}^{(k+1)}, \mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})\\
&\le E(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) + H^{(k)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})\\
&= \alpha\|\mathbf{f}\|_2^2 + \lambda\Big(\|\mathbf{b}_1^{(k)} + \mathbf{f}\|_2^2 + \|\mathbf{b}_2^{(k)}\|_2^2 + \|\mathbf{b}_3^{(k)}\|_2^2\Big)\\
&\le (\alpha + 2\lambda)\|\mathbf{f}\|_2^2 + \lambda\Big(2\|\mathbf{b}_1^{(k)}\|_2^2 + \|\mathbf{b}_2^{(k)}\|_2^2 + \|\mathbf{b}_3^{(k)}\|_2^2\Big)\\
&\le (\alpha + 2\lambda)\|\mathbf{f}\|_2^2 + \frac{mn(\mu^2 + 1)}{2\lambda}
\end{aligned}$$

which is a finite value independent of k.

Therefore, $\{(\mathbf{u}^{(k)}, \mathbf{d}^{(k)}, \mathbf{x}^{(k)}, \mathbf{y}^{(k)})\}_{k\in\mathbb{N}\setminus\{0\}}$ is a bounded sequence. By Lemma 2, we conclude that the sequence $\{(\mathbf{u}^{(k)}, \mathbf{d}^{(k)}, \mathbf{x}^{(k)}, \mathbf{y}^{(k)})\}_{k\in\mathbb{N}}$ converges and thereby $\lim_{k\to\infty} \mathbf{u}^{(k)} = \tilde{\mathbf{u}}$. □

Theorem 1 provides a solid foundation of the split-Bregman iteration, Algorithm 1, for computing the solution to the model (5), which represents a denoised image.

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References

[1] Rong Qing Jia, Hanqing Zhao, and Wei Zhao. Convergence analysis of the Bregman method for the variational model of image denoising. Appl. Comput. Harmon. Anal., 27(3):367–379, 2009.